Chapter 4. Quantum Information Theory

4.1 Physics of information

4.1.1 Maxwell's demon, Szilard's engine and the second law

• The standard reply to "why an intelligent being cannot violate the second law of thermodynamics?"

⇨ In order to obtain one bit of information, an energy $E \geq k_B T \ln 2$ must be dissipated.

• The quantum theory of measurement tells us

--it is possible to measure a certain observable without destroying or dissipating the system (quantum nondemolition measurement).
--it is impossible to delete the unknown wavefunction by a unitary process (no deletion theorem).

⇩

The ultimate source of dissipation is not acquisition of the knowledge (measurement or copying) but erasure of the information.

⇩

In order to erase one bit of information, an energy $E \geq k_B T \ln 2$ must be dissipated into reservoirs. That is, clearing a memory is thermodynamically costly.

Maxwell's Demon (1871) violates

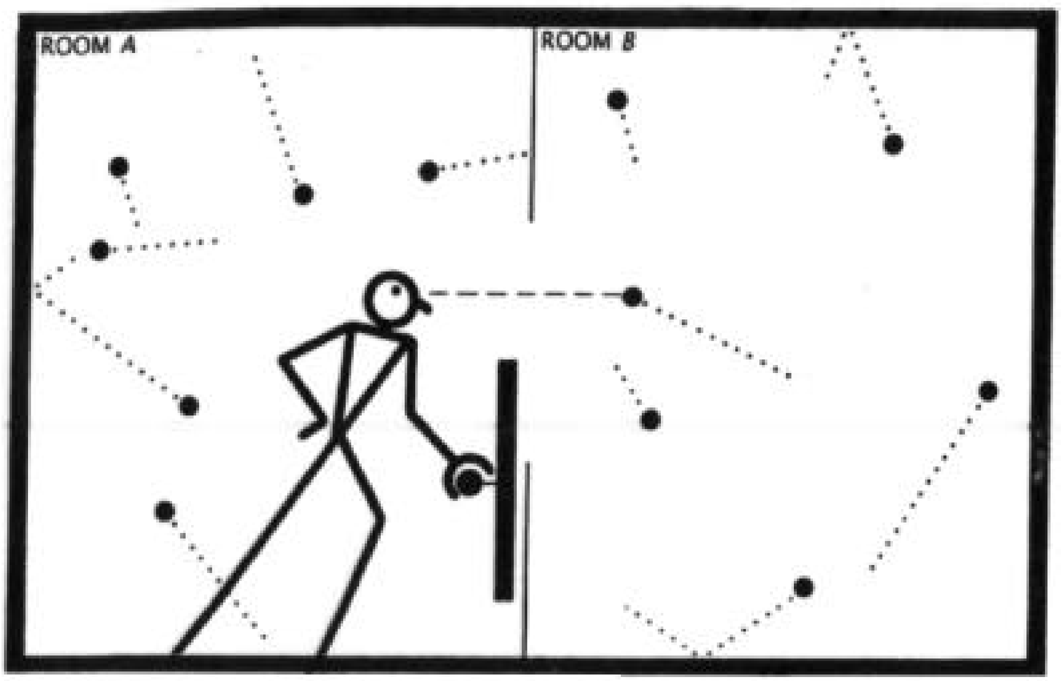
Second law of thermodynamics:

- $\Delta S \geq 0$ (reversible vs. irreversible) ←--- thermally isolated system
  example: free expansion
- $dS = dQ/T$ ←--- thermally open system

Entropy:

$S = k_B \ln \Omega$

# of accessible states



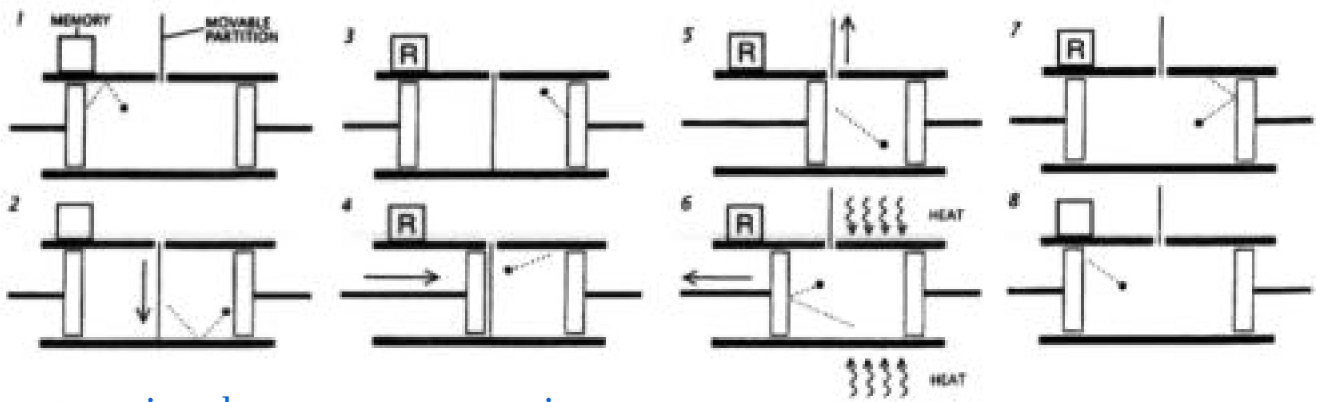
$$dS_A = \frac{dQ}{T_A} \quad (< 0) \quad + \quad dS_B = \frac{dQ}{T_B} \quad (> 0) \quad < 0$$ (entropy decrease)

Demon exorcism:

Leo Szilard
Leon Brillouin
Denis Gabor

Szilard’s Engine (1929)
¾ A single molecule machine ¾

initial state | measurement | remove partition | work done

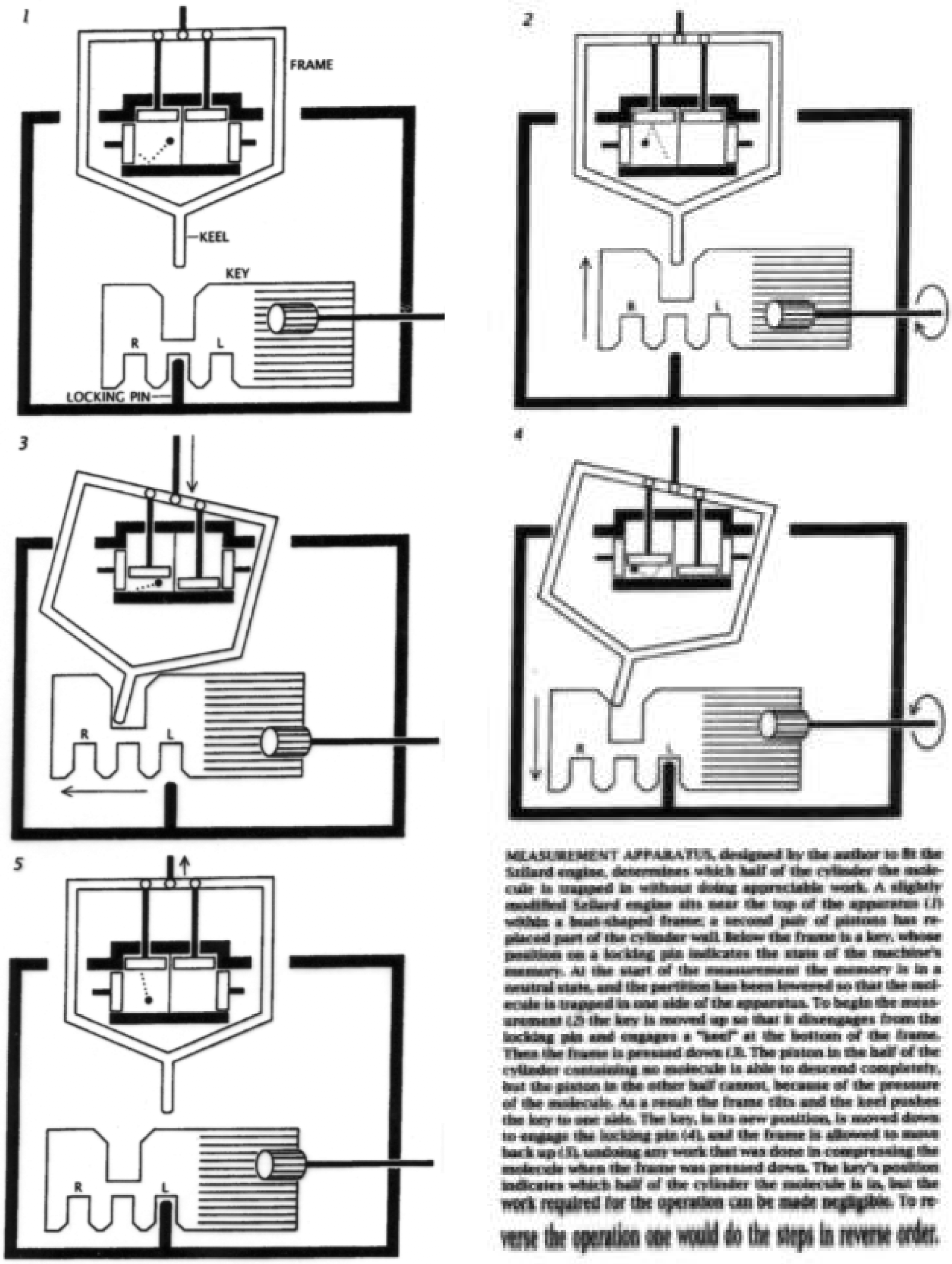


trapping the molecule | compression stroke | power stroke (work) | erase memory information

Standard reply:

A measurement costs, at least, one photon with an energy of $E \geq k_B T \ln 2$, in order to distinguish a probing photon from a thermal background radiation. The dissipation of this photon energy as a heat increases the entropy in the environment.

Bennett's Reversible Measurement (1987)



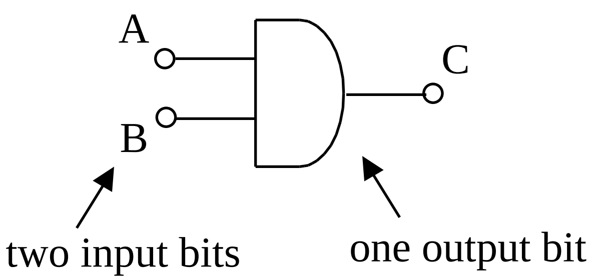
MEASUREMENT APPARATUS, designed by the author to fit the Szilard engine, determines which half of the cylinder the molecule is trapped in without doing appreciable work. A slightly modified Szilard engine sits near the top of the apparatus (1) within a boat-shaped frame; a second pair of pistons has replaced part of the cylinder wall. Below the frame is a key, whose position on a locking pin indicates the state of the machine's memory. At the start of the measurement the memory is in a neutral state, and the partition has been lowered so that the molecule is trapped in one side of the apparatus. To begin the measurement (2) the key is moved up so that it disengages from the locking pin and engages a "keel" at the bottom of the frame. Then the frame is pressed down (3). The piston in the half of the cylinder containing no molecule is able to descend completely, but the piston in the other half cannot, because of the pressure of the molecule. As a result the frame tilts and the keel pushes the key to one side. The key, in its new position, is moved down to engage the locking pin (4), and the frame is allowed to move back up (5), undoing any work that was done in compressing the molecule when the frame was pressed down. The key's position indicates which half of the cylinder the molecule is in, but the work required for the operation can be made negligible. To reverse the operation one would do the steps in reverse order.

The demon cannot violate the second law:

It must discard the past information to start a new cycle.

4.1.2 Reversible computation

The standard implementation of Boolean functions uses such primitives as AND, NOT and FAN-OUT.



Input A	Input B	Output C
0	0	0
0	1	0
1	0	0
1	1	1

AND circuit shown above is a logically irreversible gate. It is impossible to go back from the output to the input. The loss of one bit of information costs the energy $k_B T \ln 2$ at least.

A fundamental quantum dynamics is a reversible process, so there is no natural match between the standard logic primitives and future quantum logic gates.

$\Downarrow$

Is there any way to construct an arbitrary Boolean function by logically reversible gates?

Conservatic logic = # of 0's and 1's are conserved in the input and output

(1) unit wire

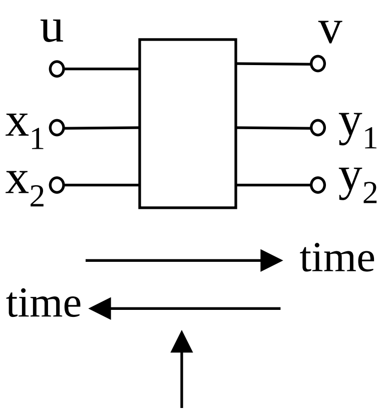
x^t	$\longrightarrow$	y^{t+1}
0		0
1		1

(one unit of time evolution)

If $x = y$ (identical position), the unit wire represents a memory.

If $x \neq y$ (different position), the unit wire represents a transmission line.

(2) Fredkin gate

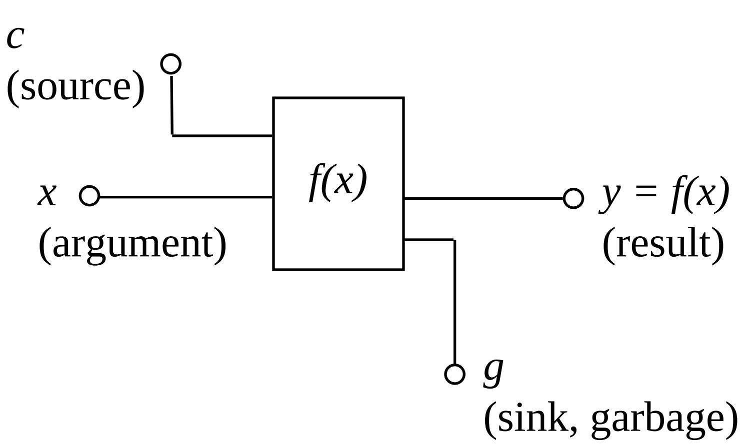


$v = u$ (control)

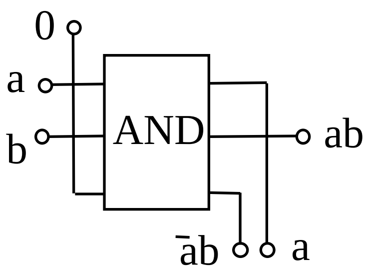
$y_1 = u x_1 + \bar{u} x_2$
$y_2 = \bar{u} x_1 + u x_2$ (conditional cross-over)

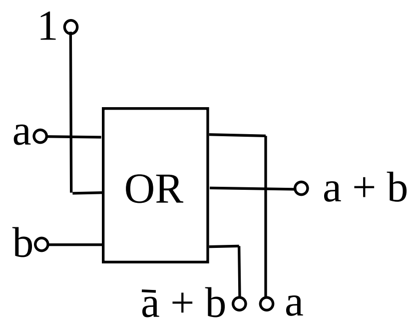
invertible logic $\Rightarrow$ natural match to reversible quantum process

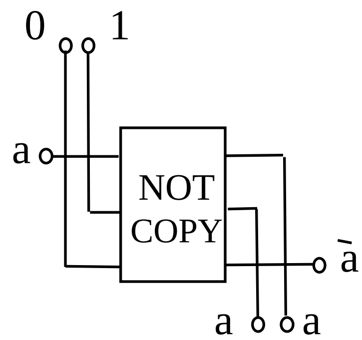
With the unit wire and the Fredkin gate, we can implement any Boolean function *f(x)* in a following configuration:



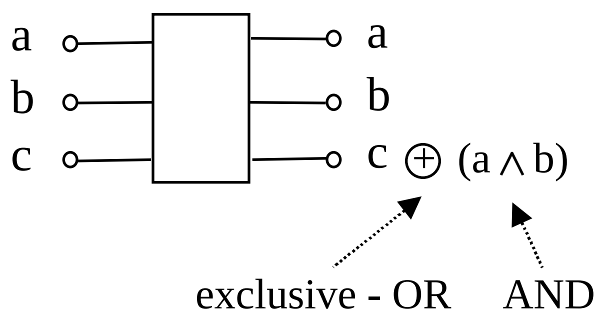
Examples:





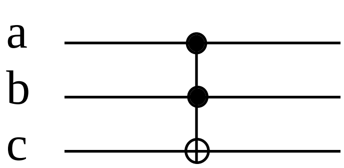


(3) Toffoli gate

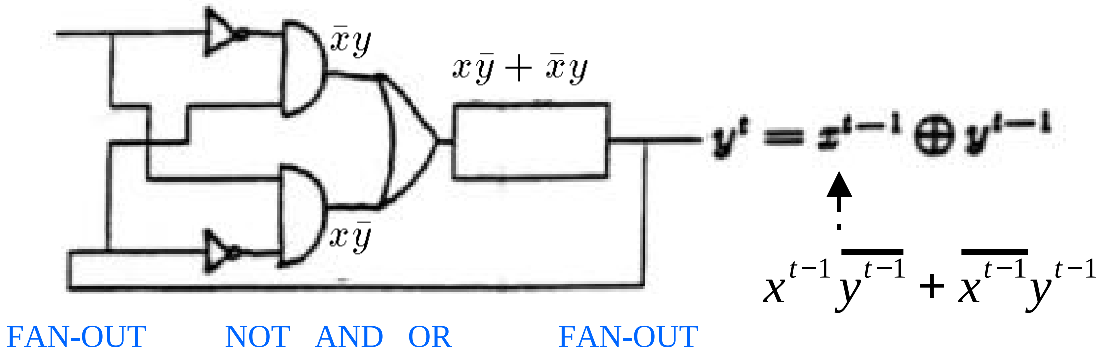


When and only when *a* and *b* are *1's*, *c* is inverted.

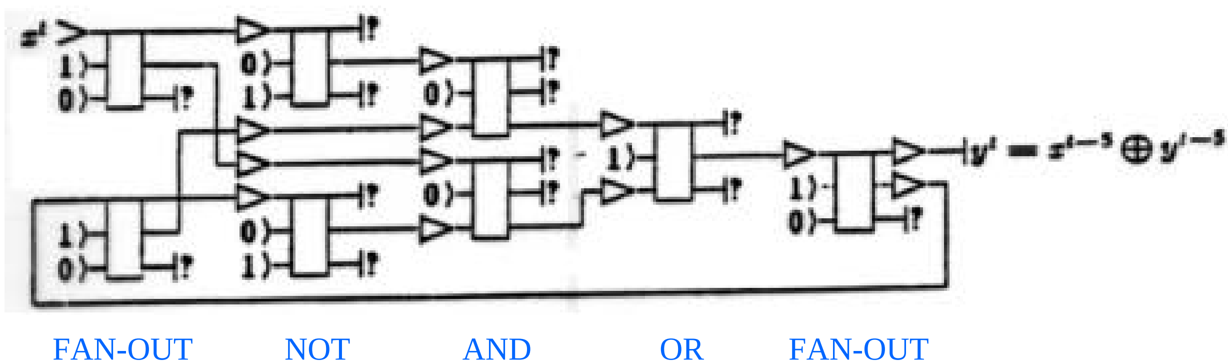
- If $c = 1$, Toffoli gate becomes NAND gate.
- Toffoli gate is sometimes referred to as Controlled-Controlled-NOT gate.



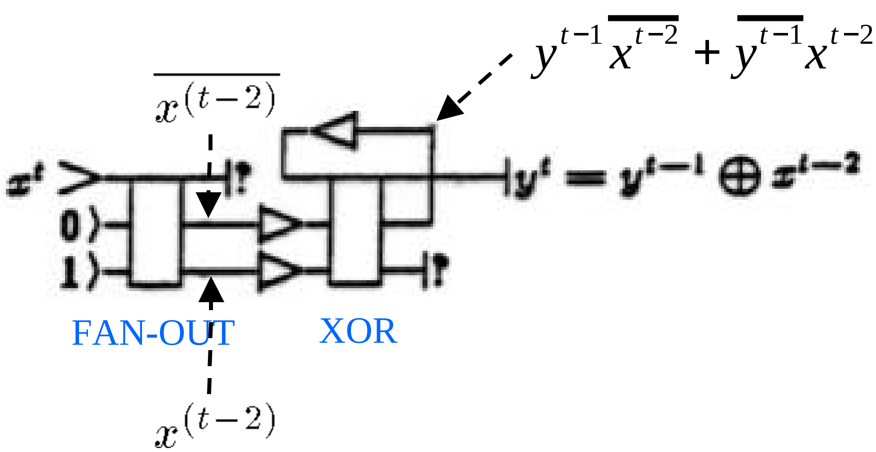
A serial adder (mod 2) by standard logic elements



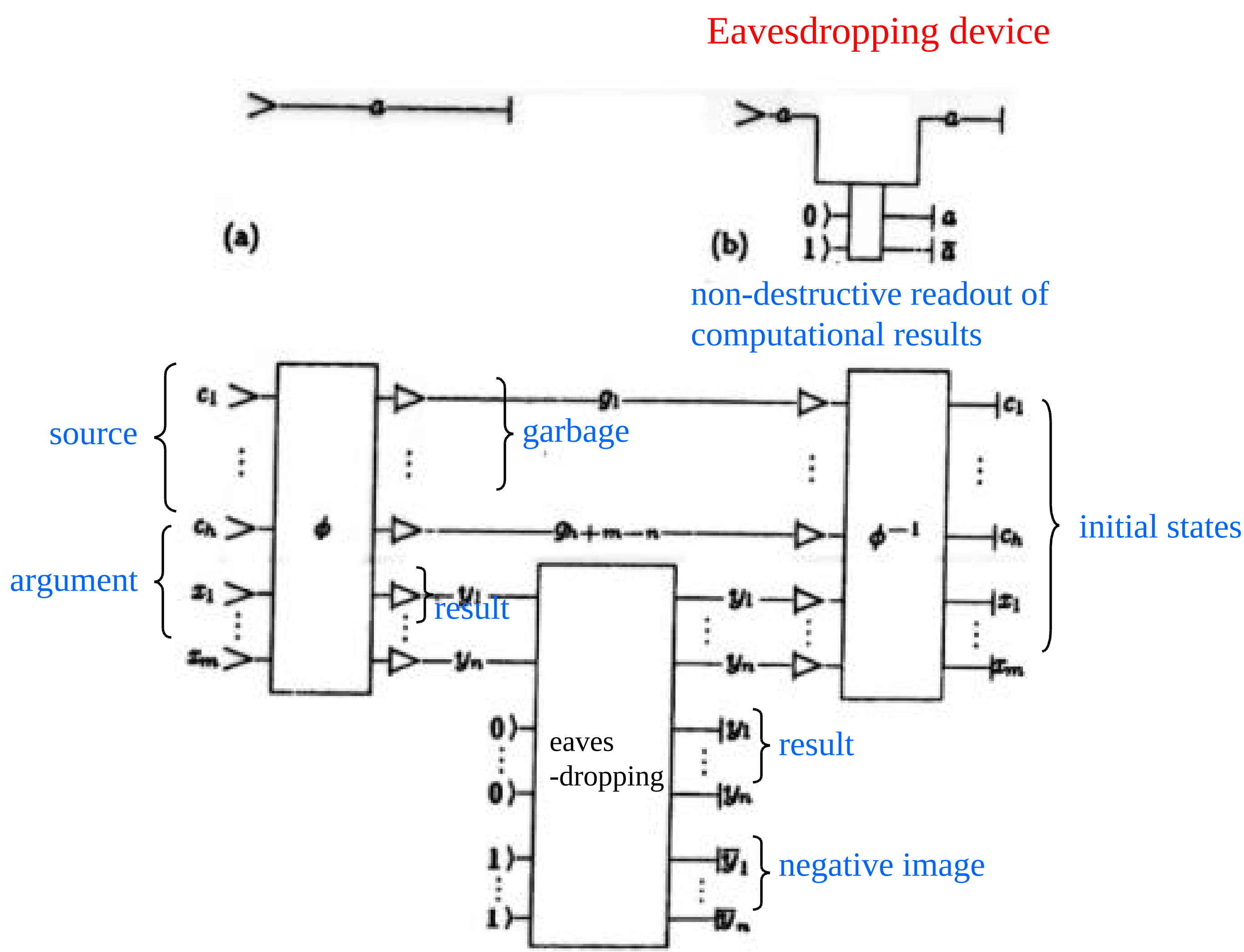
Sequential implementation by a conservatic logic circuit



Simplified implementation



Closed (Garbageless) Conservative-Logic Computers



A closed conservatic-logic computer does not increase the entropy of the computer's environment.

4.1.3 Quantum computation

A. Quantum parallelism (D. Deutsch)

Quantum bit = qubit

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \left(|a|^2 + |b|^2 = 1\right)$$

n quantum registers can represent 2^n different states simultaneously.

$$\begin{aligned}|\psi\rangle_1|\psi\rangle_2\cdots|\psi\rangle_n &= \left(a_1|0\rangle_1 + b_1|1\rangle_1\right)\ \left(a_2|0\rangle_2 + b_2|1\rangle_2\right)\ \cdots\ \left(a_n|0\rangle_n + b_n|1\rangle_n\right)\\ &= a_1a_2\cdots a_n|0\rangle_1|0\rangle_2\cdots|0\rangle_n\\ &+ a_1a_2\cdots b_n|0\rangle_1|0\rangle_2\cdots|1\rangle_n\\ &\ \ \vdots\\ &+ b_1b_2\cdots b_n|1\rangle_1|1\rangle_2\cdots|1\rangle_n\end{aligned}$$

1st 2nd *n*-th registor

$$\left.\begin{aligned}|0\rangle_L &= |00\cdots\cdots 0\rangle\\ |1\rangle_L &= |00\cdots\cdots 1\rangle\\ &\ \vdots\\ |2^n-1\rangle_L &= |11\cdots\cdots 1\rangle\end{aligned}\right\} \ 2^n \text{ different states}$$

input state for each register: $|\psi_i\rangle = \dfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$\Downarrow$

$$|\psi\rangle = |\psi_1\rangle|\psi_2\rangle\cdots|\psi_n\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=1}^{2^n-1}|x\rangle$$

Example: $n = 100$ quantum registers

$\Downarrow$

$2^{100} \simeq 10^{30}$ different input states

$\Downarrow$

$|\psi\rangle_{in} \rightarrow \boxed{\hat{U}} \rightarrow |\psi\rangle_{out}$ simultaneous computation of $\sim 10^{30}$ different input states

$\Downarrow$ equivalent

$10^{30} \times 10^{30}$ ----- beyond the capability of any fore seeable classical computer

matrix calculation

B. Quantum simulation (R. P Feynman)

Bell's theorem: No classical hidden variable theory can reproduce the prediction of quantum mechanics for certain problems.

$\Downarrow$

A certain quantum system cannot be simulated by *n*-bit classical computers with a probabilistic algorithm.

Density matrix master equation $\longrightarrow$ $2^n \times 2^n$ matrix elements
Quantum Monte-Carlo wavefunction $\longrightarrow$ 2^n vectors